Random wavelet series - Part2

Béatrice VEDEL

Université de Bretagne Sud

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Definition

A stochastic process $B = \{B(t),\, t \geq 0\}$ with real values is a brownian motion if

1. $B(0) = 0$ almost surely
2. The increments of B are independent
3. B has stationary increments and $B(t) - B(s) \simeq \mathcal{N}(0, t-s)$ for all $t > s \geq 0$
4. Sample paths of B are almost surely continuous.

Theorem (Construction via the Schauder basis)

Let ψ the Haar wavelet and Λ defined as its primitive

$$\Lambda(x) = \int_0^x \psi(t)dt = \begin{cases} x & \textit{if } 0 \leq x < 1/2 \\ 1 - x & \textit{if } 1/2 \leq x < 1 \\ 0 & \textit{otherwise} \end{cases}$$

Let $\chi, \chi_{j,k},\ j \in \mathbb{N},\ k \in \{0, ..., 2^j - 1\}$ I.I.D. random variables of law $\mathcal{N}(\prime, \infty)$. For all $t \in [0,1]$, one sets

$$B(t) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \chi_{j,k} 2^{-j/2} \Lambda(2^j t - k) + \chi t.$$

Then $B = \{B(t),\ 0 \leq t \leq 1\}$ is a brownian motion on $[0,1]$.

Proposition

The Brownian motion is self-similar, that is

$$B_{at} \overset{fdd}{=} a^{1/2} B_t$$

and its covariance operator is

$$K(t,s) = \min(s,t) = \frac{1}{2}[s+t-|t-s|]$$

Theorem (Regularity of the sample paths)

The sample paths of the brownian motion belong locally a.s. to $\mathcal{C}^{1/2-\epsilon}$ for all $\epsilon > 0$.

Theorem (Irregularity of the sample paths)

The sample paths of the brownian motion a.s. do not belong locally to $\mathcal{C}^{1/2}$.

Definition

The fractional Brownian motion (fBm) is defined equivalently, for $H \in (0,1)$ by

1. B^H is a centered Gaussian process such that

$$\forall s,t \in \mathbb{R}_+,\ K(s,t) = \frac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}]$$

2. B^H is defined by its **harmonizable representation**

$$B_t^H = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}} \widehat{W}(d\xi)$$

3. B_H is the unique self-similar gaussian process with stationary increments

$$B_{at}^H \overset{f.d.d.}{=} a^H B_t, \quad \forall a > 0, t \in \mathbb{R}_+$$

1. Decomposition in vaguelets of the fractional brownian motiin : Meyer, Sellan, Taqqu

2. Regularity and irregularity properties for fBm : it is monofractal of value $H - \epsilon$.

3. Recent results on slow and rapid points : Esser and Loosveldt (2022)

Extension of Kahane's result on the bronwian motion : Almost surely,

- almost every $t \in [0,1]$ is an ordinary points, i.e.

$$0 \leq \limsup_{s \to t} \frac{|X(t) - X(s)|}{|t-s|^H \sqrt{\log(\log(|t-s|^{-1}}} < +\infty$$

- There exists a dense set of rapid points $t \in [0,1]$, i.e.

$$0 \leq \limsup_{s \to t} \frac{|X(t) - X(s)|}{|t-s|^H \sqrt{\log(|t-s|^{-1}}} < +\infty$$

- There exists a dense set of slow points $t \in [0,1]$, i.e.

$$0 \leq \limsup_{s \to t} \frac{|X(t) - X(s)|}{|t-s|^H} < +\infty$$

Two gaussian self-similar extensions

- Fractional Brownian field Y^H of Hurst index $H \in (0,1)$ (Levy fBm)
  - covariance function

$$\mathbb{E}(Y^H_{\mathbf{x}} Y^H_{\mathbf{y}}) = C_{N,H} \left(\|\mathbf{x}\|^{2H} + \|\mathbf{y}\|^{2H} - \|\mathbf{x}-\mathbf{y}\|^{2H} \right)$$

  - harmonizable representation

$$Y^H_{\mathbf{x}} = \int_{\mathbb{R}^N} \frac{e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} - 1}{\|\boldsymbol{\xi}\|^{H+\frac{N}{2}}} d\widehat{\mathbf{W}}(\boldsymbol{\xi})$$

  - isotropic self-similar field with stationary increments

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  - isotropic self-similar field with stationary increments
- Fractional Brownian sheet $S^{\mathbf{H}}$ of Hurst index $\mathbf{H} = (H_1, ..., H_N) \in (0,1)^N$
  - covariance function
$$\mathbb{E}(S^{\mathbf{H}}_{\mathbf{x}} S^{\mathbf{H}}_{\mathbf{y}}) = \prod_{m=1}^{N} \frac{1}{2} \left(|x_m|^{2H_m} + |y_m|^{2H_m} - |x_m - y_m|^{2H_m} \right)$$
  - harmonizable representation
$$S^{\mathbf{H}}_{\mathbf{x}} = \int_{\mathbb{R}^N} \prod_{m=1}^{N} \frac{e^{i x_m \xi_m} - 1}{|\xi_m|^{H_m + \frac{1}{2}}} d\widehat{\mathbf{W}}(\boldsymbol{\xi})$$
  - non-isotropic self-similar field with stationary rectangular increments

Rectangular increments of a field X

$$\Delta X_{(x_1,x_2);(y_1,y_2)} := X_{(x_1+y_1,x_2+y_2)} - X_{(y_1,x_2+y_2)} - X_{(x_1+y_1,y_2)} + X_{(y_1,y_2)}$$

Stationary rectangular increments

$$\{\Delta X^{\alpha,H}_{(x_1,x_2);(y_1,y_2)}\}_{(x_1,x_2)\in\mathbb{R}^2} \overset{(d)}{=} \{X^{\alpha,H}_{(x_1,x_2)}\}_{(x_1,x_2)\in\mathbb{R}^2}$$

Tensorized structure and rectangular increments

$$(e^{i(x_1+h_1)\xi_1}-1)(e^{i(x_2+h_2)\xi_2}-1)-(e^{ix_1\xi_1}-1)(e^{i(x_2+h_2)\xi_2}-1)$$
$$-(e^{i(x_1+h_1)\xi_1}-1)(e^{ix_2\xi_2}-1)+(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)$$

$$= (e^{i(x_1+h_1)\xi_1}-e^{ix_1\xi_1})(e^{i(x_2+h_2)\xi_2}-e^{ix_2\xi_2})$$

Gaussian self-similar field with stationary rectangular increments different to fBs ?

$\longrightarrow$ Makogin and Mishura (2015) : construction via a covariance function

Brownian field vs brownian sheet

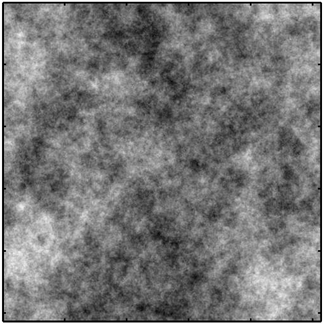
Brownian field

- Self-similar
- Stationary increments
- Harmonizable representation :

$$\mathsf{Y}^H_{\mathbf{x}} = \int_{\mathbb{R}^N} \frac{e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle}-1}{\|\boldsymbol{\xi}\|^{H+\frac{N}{2}}} d\widehat{\mathbf{W}}(\boldsymbol{\xi})$$

- Covariance function :

$$C_{N,H}\left(\|\mathbf{x}\|^{2H}+\|\mathbf{y}\|^{2H}-\|\mathbf{x}-\mathbf{y}\|^{2H}\right)$$



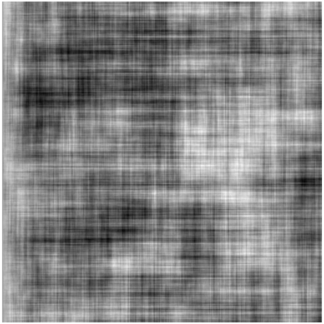
Brownian sheet

- Self-similar
- Stationary rectangular increments
- Harmonizable representation :

$$\mathsf{S}^{\mathbf{H}}_{\mathbf{x}} = \int_{\mathbb{R}^N} \prod_{m=1}^{N} \frac{e^{i x_m \xi_m}-1}{|\xi_m|^{H+\frac{1}{2}}} d\widehat{\mathbf{W}}(\boldsymbol{\xi})$$

- Covariance function :

$$\prod \frac{1}{2}\left(|x_m|^{2H}+|y_m|^{2H}-|x_m-y_m|^{2H}\right)$$



Anisotropic extensions

Operator scaling :

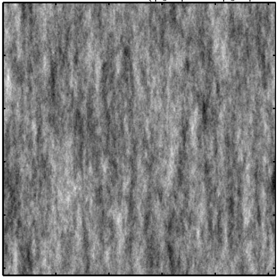
$$\forall a > 0, \quad X_{a^D \mathbf{x}} \overset{(d)}{=} a^H X(a\mathbf{x})$$

with $D = diag(\beta_1, ..., \beta_d)$ and $a^D = (a^{\beta_1}, a^{\beta_2}, ..., a^{\beta_d})$.

Operator Scaling Gaussian Fields

- Harmonizable representation

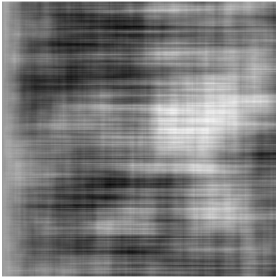
$$\mathsf{Y}_{\mathbf{x}}^{\beta,H} = \int_{\mathbb{R}^2} \frac{e^{i\langle \mathbf{x},\xi\rangle} - 1}{(|\xi_1|^{\frac{2}{\beta}} + |\xi_2|^{\frac{2}{(2-\beta)}})^{H+1}} d\widehat{\mathbf{W}}(\xi)$$



Multifractional brownian sheet

- Harmonizable reprentation

$$\mathsf{Y}_{\mathbf{x}}^{H_1,H_2} = \int_{\mathbb{R}^2} \prod_{m=1}^{2} \frac{e^{ix_m\xi_m} - 1}{|\xi_m|^{H_m+\frac{1}{2}}} d\widehat{\mathbf{W}}(\boldsymbol{\xi})$$



Function spaces

- Sobolev, Besov spaces in 1D :
- Two extensions in $\mathbb{R}^d$:

Classical spaces

- $f \in H^1(\mathbb{R})$ iff
$f, \frac{\partial f}{\partial_x}, \frac{\partial f}{\partial_y} \in L^2(\mathbb{R}^2)$
- Definitions in the Fourier domain via the Littlewood-Paley Analysis
- Classical Wavelet Analysis in $\mathbb{R}^d$
- Anistropic function spaces

space with Dominating mixed smoothness

- $f \in SH^1(\mathbb{R})$ iff
$f, \frac{\partial f}{\partial_x}, \frac{\partial f}{\partial_y}, \frac{\partial^2 f}{\partial x \partial y} \in L^2(\mathbb{R}^2)$
- Definition in the Fourier domain via the Hyperbolic Littlewood-Paley Analysis
- Hyperbolic Wavelet Analysis
- Anisotropic function spaces with DMS (Triebel, Schmeisser, Vybiral, Ullrich)

Wavelet Analysis

In 1-D : Two functions φ and ψ such that

$$\{\varphi(\cdot - k),\ k \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j \cdot -k),\ j \geq 0,\ k \in \mathbb{Z}\}$$

Classical 2D Wavelet Analysis

- The systems, with $k_1, k_2 \in \mathbb{Z}$

$\{\varphi(x_1 - k_1)\varphi(x_2 - k_2)\}$
$\{2^j\varphi(2^jx - k_1)\psi(2^jx - k_2),\ j \geq 0\}$
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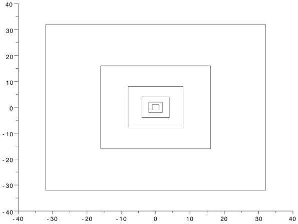
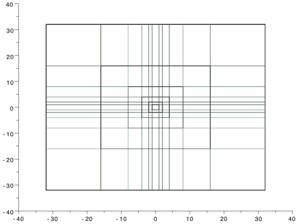


Hyperbolic wavelet analysis

- The systems, with $k_1, k_2 \in \mathbb{Z}$

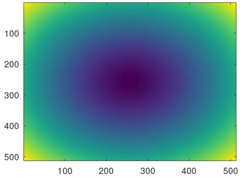
$\{\varphi(x_1 - k_1)\varphi(x_2 - k_2)\}$
$\{2^{\frac{j_1+j_2}{2}}\psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2),\ j_1, j_2 \geq 0\}$

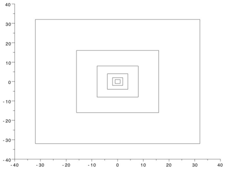


Wavelet Analysis

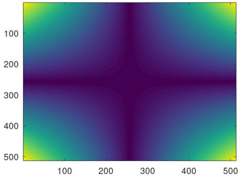
How the spectral density behaves compared to the wavelet filtering ?

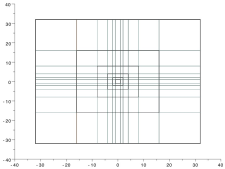
Brownian field





Brownian sheet





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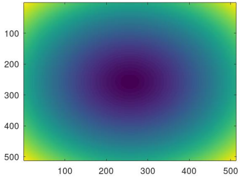
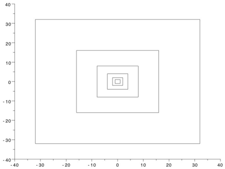
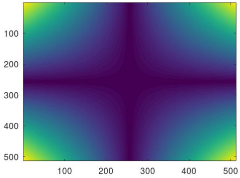
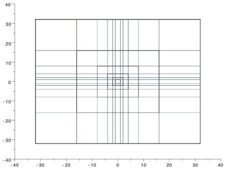




Wav. characterizations of the Brownian field

Brownian sheet





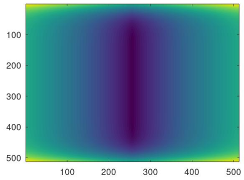
Hyperbolic Wav. characterizations of the Brownian Sheet (Ayache, Xiao, 2004)

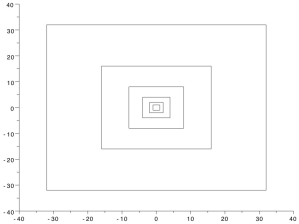
$\Rightarrow$ Estimation of the Hurst parameter

Wavelet Analysis

And for the anisotropic setting ?

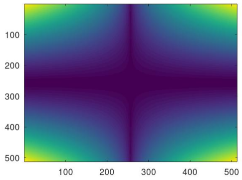
Brownian field

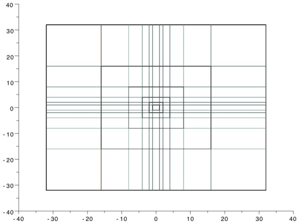




Wav. characterizations of the Brownian field

Brownian sheet

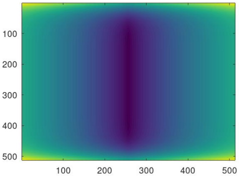


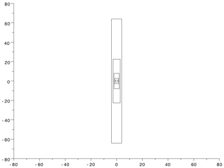


Hyperbolic Wav. characterizations of the

Anisotropic Analysis

Replace with anisotropic wavelet basis (Triebel)

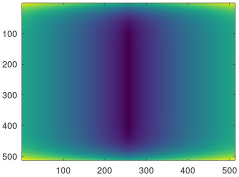


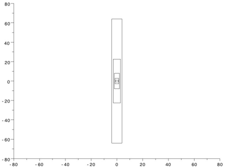


$\Rightarrow$ Estimation of the Hurst parameter when the anisotropy is known.

Anisotropic Analysis

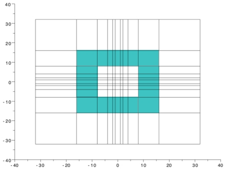
Replace with anisotropic wavelet basis (Triebel)

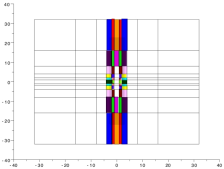


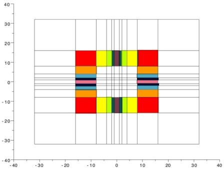


⇒ Estimation of the Hurst parameter when the anisotropy is known.

Hyperbolic wavelet transforms can ”almost” characterize classical and anistropic Besov spaces







⇒ Hypbolic wav. characterizations of Brownian field and of Operator scaling Gaussian

Hyperbolic wavelet analysis

- Hyperbolic wavelet basis :

$$\psi_{j_1,j_2,k_1,k_2}(x_1,x_2) = \psi(2^{j_1}x_1 - k_1)\psi(2^{j_2}x_2 - k_2), \quad j_1, j_2 \geq 0,\ k_1, k_2 \in \mathbb{Z}$$

(DeVore, Konyagin, Temlyakov, 98)
'Rectangular supports' of size $2^{-j_1} \times 2^{-j_2}$

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- Hyperbolic wavelet coefficients of f

$$c_{j_1,j_2,k_1,k_2} = 2^{j_1+j_2}\langle f,\, \psi_{j_1,j_2,k_1,k_2}\rangle$$

Hyperbolic wavelet analysis

Theorem (work with P. Abry, M. Clausel, S. Jaffard, S. Roux, and with M. Schäfer, T. Ullrich)

- If $f \in \mathcal{C}^{s,\beta}(\mathbb{R}^2)$ then

$$\forall (j_1, j_2) \quad \sup_{k_1,k_2} |c_{j_1,j_2,k_1,k_2}(f)| \leq C 2^{-\max\left(\frac{j_1}{\beta}, \frac{j_2}{2-\beta}\right)s} \quad (\star)$$

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- General version of this result for anisotropic Besov spaces :

$$f \in B^{s,\beta}_{p,q} \overset{\log}{\simeq} \sum_{(j_1,j_2)\in\mathbb{N}_0^2} 2^{\max(\frac{j_1}{\beta}, \frac{j_2}{2-\beta})sq} 2^{-\frac{(j_1+j_2)q}{p}} \|c_{j_1,j_2,\cdot,\cdot}\|^q_{\ell^p} < +\infty \ .$$

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- And for Triebel-Lizorkhin spaces.

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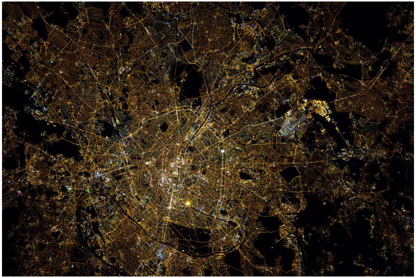
- And for Triebel-Lizorkhin spaces.
- The wavelet characterizations of spaces of dominating mixed smoothness involves a sum instead of a max

Weighted tensorized textures

New textures (and new notions of smoothness) between brownian field and brownian sheet.



Motivation



Motivation

Introduce intermediate “tensorized” gaussian textures (dim 2)

- yielding the fractional Brownian sheet for $\alpha = 0$ and a field similar to the fractional Brownian field in terms of regularity for $\alpha = 1$.
- keeping some fundamental properties of self-similarity and rectangular stationary increments
- with an harmonizable representation

Fix $\alpha \in [0,1]$, $H \in (0,1)$ and set

$$H_\alpha^+ := (1+\alpha)H \qquad \text{and} \qquad H_\alpha^- := (1-\alpha)H$$

We define the weighted tensorized fractional brownian field (WTFBF) $X^{\alpha,H}$ by

$$X_{(x_1,x_2)}^{\alpha,H} := \int_{\mathbb{R}^2} \frac{(e^{ix_1\xi_1}-1)(e^{ix_2\xi_2}-1)}{\min\{|\xi_1|,|\xi_2|\}^{H_\alpha^-+\frac{1}{2}} \max\{|\xi_1|,|\xi_2|\}^{H_\alpha^++\frac{1}{2}}} d\widehat{\mathbf{W}}(\xi_1,\xi_2)$$

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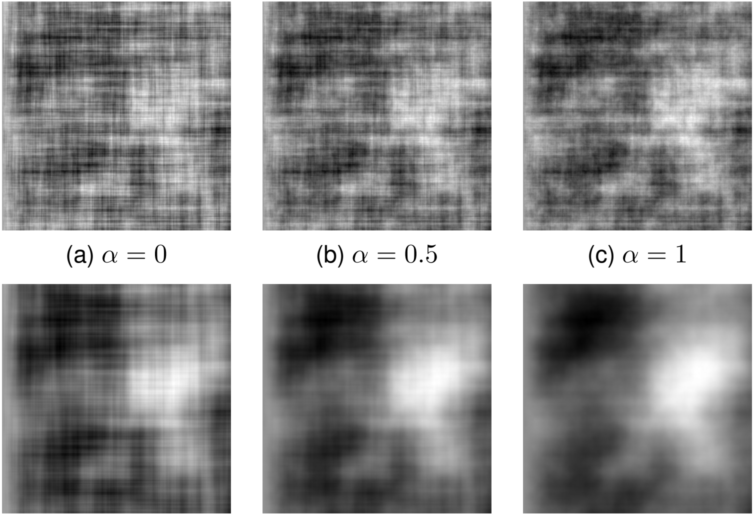
- the field is well-defined since its kernel $\mathcal{K}_{(x_1,x_2)}^{\alpha,H}$ belongs to $L^2(\mathbb{R}^2)$.
- the Fourier transform of $\mathcal{K}_{(x_1,x_2)}^{\alpha,H}$ is real, and it implies that the field is real since

$$X_{(x_1,x_2)}^{\alpha,H} = \int_{\mathbb{R}^2} \mathcal{K}_{(x_1,x_2)}^{\alpha,H}(\xi_1,\xi_2) d\widehat{\mathbf{W}}(\xi_1,\xi_2) = \int_{\mathbb{R}^2} \widehat{\mathcal{K}_{(x_1,x_2)}^{\alpha,H}}(\xi_1,\xi_2) d\mathbf{W}(\xi_1,\xi_2)$$

- self-similar of parameter $2H$
- stationary rectangular increments, stationary horizontal and vertical increments



(a) $\alpha = 0$ (b) $\alpha = 0.5$ (c) $\alpha = 1$

(d) $\alpha = 0$ (e) $\alpha = 0.5$ (f) $\alpha = 1$

Figure – Weighted tensorized fractional Brownian fields with parameters $H = 0.3$ (a-c) and $H = 0.7$ (d-f) simulated using a spectral representation approximation method

- Computation of the variance of the rectangular increments
- Variant of Kolmogorov theorem
- Regularity of sample paths
- Irregularity of sample paths using a hyperbolic Meyer wavelet analysis

Hyperbolic Littlewood-Paley analysis

- Let $\theta_0 \in \mathcal{S}(\mathbb{R})$ be a non-negative function supported on $[-2,2]$ with $\theta_0 = 1$ on $[-1,1]$. For any $j \geq 1$, we define

$$\theta_j = \theta_0(2^{-j}\cdot) - \theta_0(2^{-(j-1)}\cdot)$$

so that $\sum_{j\geq 0} \theta_j = 1$ and $\operatorname{supp}(\theta_j) \subset \{\xi \in \mathbb{R} : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

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- For any $\bar{j} = (j_1, j_2) \in \mathbb{N}_0^2$ and any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ set

$$\theta_{\bar{j}}(\xi_1, \xi_2) := \theta_{j_1}(\xi_1)\theta_{j_2}(\xi_2) \ .$$

The function $\theta_{\bar{j}}$ belongs to $\mathcal{S}(\mathbb{R}^2)$ for all $\bar{j} \in \mathbb{N}_0^2$, is compactly supported on a dyadic rectangle and $\sum_{\bar{j}\in\mathbb{N}_0^2} \theta_{\bar{j}} = 1$.

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- For $f \in \mathcal{S}'(\mathbb{R}^2)$, we set

$$\Delta_{\bar{j}} f := \mathcal{F}^{-1}(\theta_{\bar{j}} \mathcal{F} f) \ .$$

The sequence $(\Delta_{\bar{j}} f)_{\bar{j}\in\mathbb{N}_0^2}$ is called a hyperbolic Littlewood-Paley analysis of f.

Definition

For $p, q \in (0, +\infty]$, $s \in \mathbb{R}$ and $\alpha \in [0,1]$, we define the weighted tensorized Besov space $T^{s,\alpha}_{p,q}B(\mathbb{R}^2)$ via

$$T^{s,\alpha}_{p,q}B(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) \ : \ \|f\|_{T^{s,\alpha}_{p,q}B} < \infty \right\},$$

where

$$\|f\|_{T^{s,\alpha}_{p,q}B} := \left(\sum_{\bar{j} \in \mathbb{N}_0^d} 2^{((1+\alpha)\max(\bar{j}) + (1-\alpha)\min(\bar{j}))sq} \|\Delta_{\bar{j}} f\|_p^q \right)^{\frac{1}{q}}$$

with the usual modification in case $q = \infty$.

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with the usual modification in case $q = \infty$.

- for $\alpha = 0$, the space is the Besov space of dominating mixed smoothness $S^s_{p,q}B(\mathbb{R}^2)$.
- for $\alpha = 1$, the space is the hyperbolic Besov space $\widetilde{B}^{2s}_{p,q}(\mathbb{R}^2)$.
- The space does not depend on the hyperbolic resolution of unity. Moreover, θ_0 can be chosen with an arbitrary compact support.

Choose θ_0 as the Fourier transform of a Meyer scaling function $\theta_0 = \widehat{\varphi}$.

Theorem

Let $p, q \in (0, +\infty]$, $s \in \mathbb{R}$ and $\alpha \in [0, 1]$. Then, $f \in T^{s,\alpha}_{p,q}B(\mathbb{R}^2)$ if and only if

$$\|c_{\bar{j},\bar{k}}(f)\|_{t^{s,\alpha}_{p,q}b}$$

$$:= \left(\sum_{\bar{j} \in (\mathbb{N}_0 \cup \{-1\})^2} 2^{-\frac{(j_1+j_2)q}{p}} 2^{-((1+\alpha)\max(\bar{j}) + (1-\alpha)\min(\bar{j}))sq} \left(\sum_{\bar{k} \in \mathbb{Z}^2} |c_{\bar{j},\bar{k}}(f)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < +\infty$$

Moreover, there exist two constants A and B such that

$$A \|c_{\bar{j},\bar{k}}(f)\|_{t^{s,\alpha}_{p,q}b} \leq \|f\|_{T^{s,\alpha}_{p,q}B} \leq B \|c_{\bar{j},\bar{k}}(f)\|_{t^{s,\alpha}_{p,q}b}$$

Theorem

The Meyer wavelet basis is an unconditional basis of $T^{s,\alpha}_{p,q}B(\mathbb{R}^2)$.

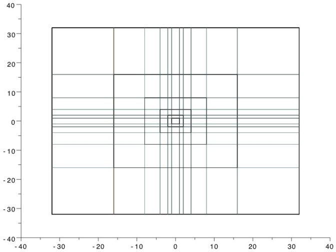
Proposition

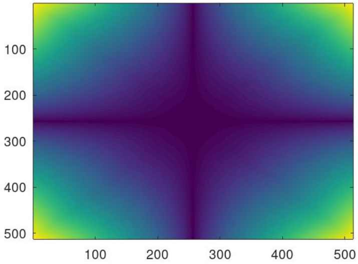
For $s \in \mathbb{R}$*,* $p, q \in (0, +\infty]$ *and* $\alpha \in [0, 1]$*, one has*

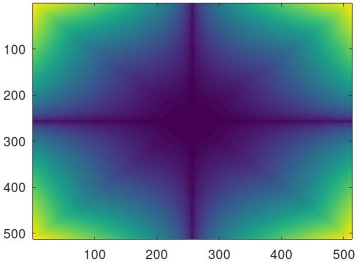
- $S^{(1+\alpha)s}_{p,q}B(\mathbb{R}^2) \hookrightarrow T^{s,\alpha}_{p,q}B(\mathbb{R}^2) \hookrightarrow S^{s}_{p,q}B(\mathbb{R}^2)$
- $\widetilde{B}^{2s}_{p,q}(\mathbb{R}^2) \hookrightarrow T^{s,\alpha}_{p,q}(\mathbb{R}^2) \hookrightarrow \widetilde{B}^{(1+\alpha)s}_{p,q}(\mathbb{R}^2)$
- $B^{2s}_{p,q,log\beta_1}(\mathbb{R}^2) \hookrightarrow T^{s,\alpha}_{p,q}(\mathbb{R}^2) \hookrightarrow B^{(1+\alpha)s}_{p,q,log\beta_2}(\mathbb{R}^2)$

Moreover, these embeddings are optimal at fixed p, q*.*

Perspectives







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Thank you for your attention