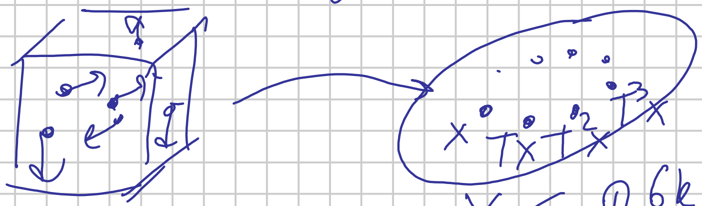
Ergodic Theorems

Summer School „Applied Analysis"
Chemnitz, Sept. 2024

1 Classical Erg. Thms

Origin ~1880, Ludwig Boltzmann, statistical mechanics,

ideal gas
k particles

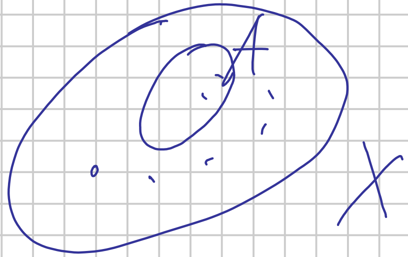


$X \subset \mathbb{R}^{6k}$
state space

$T: X \to X$
volume preserving

Boltzmann's ergodic hypothesis:

„time mean = space mean"



$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1\!\!1_A(T^n x) = \int 1\!\!1_A \, d\mu$$
rescaled volume

$\boxed{\text{Def.}}$ 1) A measure-preserving system (MDS) is (X,μ,T), where:

- (X,μ) prob. space

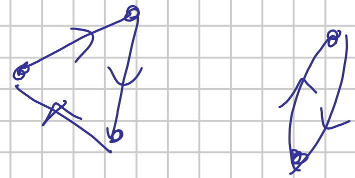
- $T : X \to X$ **μ-preserving**, i.e., $\mu(T^{-1}(A)) = \mu(A)$ $\forall A$ meas.

2) (X, μ, T) is **ergodic** if $\forall A \subset X$ meas.

A $\underbrace{T\text{-inv.}}_{T^{-1}A \subset A \text{ up to a null set}} \iff \mu(A) \in \{0, 1\}$

$\boxed{\text{Ex}}$

1) Finite systems: X finite
μ rescaling count. measure
T bij.



2) **Bernoulli shifts**: $k \in \mathbb{N}$ $X = \{0, \dots, b-1\}^{\mathbb{N}}$, $T \leftarrow$

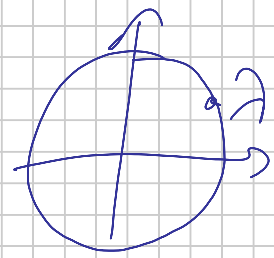
μ product measure corr. to (any) prob. measure p on $\{0, \dots, b-1\}$

$$\mu(\underbrace{X \times \dots \{a_1\} \times X \times \dots \times \{a_2\}, \dots}_{\text{cylinder}}) := p(a_1) \cdots p(a_j)$$

$(X, \mu, \leftarrow)$ is **ergodic** (even mixing: $\mu(T^{-n}(A) \cap B) \to \mu(A)\mu(B)$)

3) $(\mathbb{T}, \text{Leb}, \lambda)$ $\quad Tz := \lambda z$

unit circle $\quad$ rescaled



MDS, erg. $\Leftrightarrow$ λ irrat.

exerc. $\quad \lambda^n \neq 1 \ \forall n$

More generally: rotations on compact groups w.r.t. Haar measure

unique rotation inv. measure

4) (X, T) topological dynam. system (TDS)

comp. top. space $\quad T: X \to X$ cont.

Krylov - Bogolyubov : $\exists$ T-inv. prob. measure μ on X

$\rightsquigarrow (X, \mu, T)$ MDS

if μ is unique, (X, T) is called uniquely ergodic

Exerc.: $(\mathbb{T}, \lambda)$, λ irrat. is uniq. erg.

Linearisation

$A \rightsquigarrow \mathbb{1}_A$

$(X,\mu) \rightsquigarrow L^p(X,\mu)$, $p \in [1,\infty)$

$T \rightsquigarrow$ Koopman operator $T : L^p \to L^p$

$$(Tf)(x) = f(Tx)$$

T is:

- isometry $\|Tf\| = \|f\|$ $\forall f$
- positive, $T\mathbb{1} = \mathbb{1}$, $T\mathbb{1}_A = \mathbb{1}_{T^{-1}(A)}$

  $Tf \geq 0$ for $f \geq 0$
- invert. if invert. transf.

Rem. (X,μ,T) erg. $\iff$ $\mathrm{Fix}\,T = \langle \mathbb{1} \rangle$

$f : Tf = f$

MET

$\boxed{\text{Thm.}}$ (Mean ergodic thm, von Neumann '31)

$\forall H$ Hilbert $\forall T \in \mathcal{L}(H)$ isometry $\forall f \in H$

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \to P_{\mathrm{Fix}\,T} f.$$

orth. proj.

Proof (sketch, Riesz)

Step 1 von Neumann decomp.

$$H = \mathrm{Fix}\,T \overset{\perp}{\oplus} \overline{Rg(I-T)}$$

Step 2 $f \in \mathrm{Fix}\,T$ clear

$f \in Rg(1-T)$, $f = g - Tg$

$$\frac{1}{N}\sum_{n=1}^{N}(T^n g - T^{n+1} g) = \frac{Tg - T^{N+1} g}{N} \to 0$$

+ appr. arg. ∎

$\boxed{\text{Thm}}$ (Pointwise ET, Birkhoff '31)

$\forall$ MDS (X,μ,T) $\forall f \in L^1(X,\mu)$

$$\frac{1}{N}\sum_{n=1}^{N} T^n f$$

conv. a.e. to $\underset{\text{cond. expect.}}{\mathbb{E}}(f \mid \underbrace{\Sigma_{inv.}}_{\{A\ T\text{-inv.}\}})$ and

$\lim = \int f \, d\mu \cdot \mathbb{1}$ for erg. systems.

$f = \mathbb{1}_A \longrightarrow$ Proof of Boltzmann's erg. hypothesis for erg. systems and a.e. x

Rem. 1) Birkhoff + ergodicity of Bernoulli shifts imply that (exer.)

a.e. $x \in [0,1]$ is normal, i.e.,

$x = 0,a_1 a_2 \ldots$, $a_j \in \{0,\ldots,9\}$ s.t.

$\forall d \quad \forall w_1 \ldots w_d \in \{0,\ldots,9\}^d$

$$\frac{\#\{n \in \{1,\ldots,N\} : a_n = w_1, \ldots, a_{n+d-1} = w_d\}}{N} \longrightarrow \left(\frac{1}{10}\right)^d$$

$\ldots \underbrace{\ldots\ldots}_{n}\ldots$

(even $\forall$ base) Ex: $0,123\ldots 9101112\ldots$

$\sqrt{2}$, $\log 2$, e, π

2) One can show:

PET $\longleftrightarrow$ strong law of large numbers

MET $\longleftrightarrow$ weak — " —

3) <u>Birkhoff for uniquely erg. systems</u>

$\forall (X, T)$ uniq. ergodic $\forall f \in C(X)$

$$\frac{1}{N}\sum_{n=1}^{N} (T^n f)(x) \to \int f \, d\mu \quad \forall x \quad \text{even } \underline{\text{uniformly}} \text{ in } x$$

<u>Idea:</u> As in Krylov-Bog., $\forall$ weak* limit of

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{T^n x}$$

is T-inv. $\Rightarrow \mu$ (uniq. erg.), i.e.,

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \to \int f \, d\mu \quad \forall f \in C(X) \quad \forall \text{ fixed } x.$$

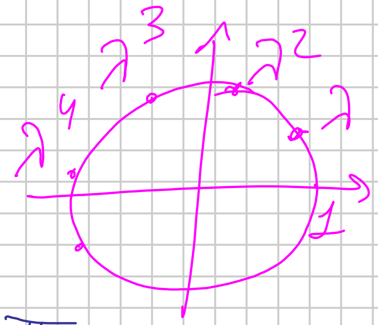
Unif.: exerc.

② Weighted Ergodic Theorems

Question/goal: Find good (mostly bdd) weights $(a_n) \subset \mathbb{C}$

s.t.
$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f$$

converge $\forall (X,\mu,T)$ $\forall f \in L^1$



Mean conv (a_n) good $\overset{\text{exer.}}{\iff}$ $\frac{1}{N}\sum_{n=1}^{N} a_n \lambda^n$ conv. $\forall \lambda \in \mathbb{T}$

(use spectral thm for $f \in L^2$) unit circle

Pointwise conv. no charact.

$\boxed{\text{Ex}}$ (λ^n) good $\forall \lambda \in \mathbb{T}$:

$Y := X \times \mathbb{T}$

$S^n(x,z) := (T^n x, \lambda^n z)$

$g(x,z) := z \cdot f(x)$

$$\frac{1}{N}\sum_{1}^{N}(S^n g)(x,z) = \frac{1}{N}\sum_{1}^{N} g(T^n x, \lambda^n z) = \underbrace{\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(T^n x)}\cdot z$$

conv. for a.e. z and x (Birkhoff)

$\boxed{\text{Thm}}$ (Wiener-Wintner ('41))

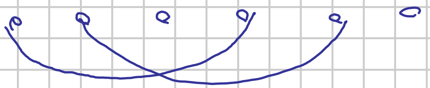
$\forall$ erg. (X,μ,T) $\forall f\in L^1$ $\exists X'\subset X$ with $\mu(X')=1$

$$\frac{1}{N}\sum_{n=1}^{N}\lambda^n f(T^n x)$$

conv. $\forall x\in X'$ $\forall \lambda\in\mathbb{T}$.

$\boxed{\text{Lemma}}$ (van der Corput)

Let $(u_n)\subset H$ bdd, denote
Hilbert



$$\gamma_k := \overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle u_{n+k}, u_n\rangle\right|$$

if $\frac{1}{N}\sum_{1}^{N}\gamma_k \longrightarrow 0$, then $\frac{1}{N}\sum_{1}^{N} u_n \longrightarrow 0$

Proof of WW (sketch)

<u>idea</u>: use decomposition

$$L^2 = \overline{\text{lin}}\{\text{eigenfcts of } T\} \oplus \{f \text{ weakly mixing}\}$$

$$f: \frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \longrightarrow 0$$

- f eigenfct: $Tf = cf$

$$\frac{1}{N}\sum_{1}^{N} \underbrace{\lambda^n c^n}_{(\lambda c)^n} f \quad ✓$$

- f weakly mixing: van der Corput:

Take $x \in X$

$$u_n := \lambda^n f(T^n x) \in \mathbb{C} \quad \text{bdd for } f \in L^\infty$$

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_{n+k} \cdot \overline{u_n}\right| = \left|\frac{1}{N}\sum_{n=1}^{N} \cancel{\lambda^{n+k}} (T^{n+k} f)(x) \cdot \cancel{\overline{\lambda}^n} (T^n \overline{f})(x)\right|$$

$$= \left|\cancel{\lambda^k} \frac{1}{N}\sum_{n=1}^{N} T^n (T^k f \cdot \overline{f})(x)\right| \xrightarrow[\substack{\text{Birkhoff} \\ \text{for } T^k f \cdot \overline{f}}]{} \left|\int T^k f \cdot \overline{f} \, d\mu\right| = \gamma_k$$

$$\frac{1}{N}\sum_{k=1}^{N} \gamma_k = \frac{1}{N}\sum_{k=1}^{N} |\langle T^k f, f\rangle| \longrightarrow 0 \text{ if } f \text{ w. mixing}$$

VdC: $\frac{1}{N}\sum_{1}^{N} u_n \to 0$

+ appr. argument for $f \in L^1$. $\blacksquare$ "

<u>Rem.</u> Proof $\Rightarrow$ uniform conv. to 0 for w. mixing fcts"

why? for uniq. erg. systems

$\boxed{\text{More examples}}$ (pointwise)

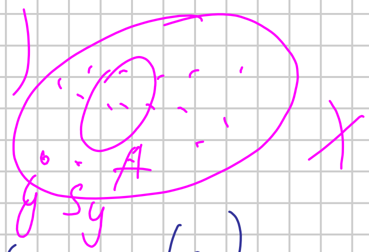
- $(\lambda^{p(n)})$, $p \in \mathbb{Z}[\cdot]$ Lesigne '93 (WW family)
- <u>return times sequences</u>:

  $\forall$ erg. (Y, ν, S) $\forall g \in L^1$, the seq.

  $$a_n := g(S^n y)$$

  is good for a.e. y (Bourgain '89)

$\left(g = 1\!\!1_A \,,\ 1\!\!1_A(S^n y) = \begin{cases} 1, & S^n y \in A \\ 0 & \text{otherw.} \end{cases} \right)$



- nilsequences (seq. coming from rotations on nilpotent groups)

  $\forall g \in C(Y)\ \ \forall y \in Y$ – also a WW-family (Host, Kra '09, E.-Zorin-Kranich '13)

- certain log-exp. fcts (many)

  Ex.: $\left(e^{2\pi i n \alpha}\right)$ good, $\left(e^{2\pi i \log n}\right)$ bad (E.-Krause '14)

- Modified von Mangoldt fct

  $\Lambda'(n) = \log n \cdot 1\!\!1_P$

  weighted ave $\longleftrightarrow$ averages along primes

$\frac{1}{N}\sum_{1}^{N} \lambda'(n) T^n f$

Everywhere conv. for nilsystems: Green-Tao '18

F. '20

$\frac{1}{N}\sum_{1}^{N} T^{p_n} f$ — p_n – primes

$\forall f \in L^p,\ p > 1$

Bourgain '88

Wierdl '88

- Möbius fct

$$\mu(n) = \begin{cases} 1, & n = p_1 \cdots p_{2k},\ p_j \text{ distinct} \\ -1, & n = p_1 \cdots p_{2k+1},\ \text{—"—} \\ 0, & n \text{ not square free} \end{cases}$$

1	2	3	4	5	6	7	8	9
1	-1	-1	0	-1	1	-1	0	0

...

$(\mu(n))$ is good with $\lim = 0$: Sarnak '11

El Abdalaoui, Kułaga, Lemańczyk, de la Rue '14

$\boxed{\text{Conjecture}}$ (Sarnak '11)

$\forall$ TDS (X,T) with topological entropy 0

$\forall f \in C(X)$ $\quad \frac{1}{N}\sum_{n=1}^{N} \mu(n)(T^n f)(x) \longrightarrow 0$

$\underline{\forall} x \in X$. Or:

„$\mu \perp$ deterministic seq."

<u>Sarnak's conj.</u>

1) holds for many systems: rotations (Davenport), nilsystems (Green-Tao '05), ...

2) has connection to <u>Chowla's conj.</u>

$$\frac{1}{N}\sum_{n=1}^{N} \mu(n+a_1)\cdot\ldots\cdot\mu(n+a_k) \longrightarrow 0$$

$\forall k \quad \forall a_1 < a_2 < \ldots < a_k$

- Chowla $\Rightarrow$ Sarnak (Sarnak '11)
- Sarnak $\Rightarrow$ Chowla for some (N_j) (Gomilko, Kwietniak, Lemańczyk '17) Tao '17 – (N_j) very large

- logarithmic versions equiv. (Tao '16)

$$\frac{1}{N}\sum_{n=1}^{N} a_n \rightsquigarrow \frac{\sum_{n=1}^{N} \frac{a_n}{n}}{\sum_{n=1}^{N} \frac{1}{n}}$$

- log. Chowla holds for
  $k=2$ (Tao '16)
  all odd k (Tao, Teräväinen '17)

For more:
book with Bálint Farkas
„A journey through ergodic thms"

$$\left|\frac{1}{N}\sum_{n=1}^{N} \mu(n)\cdot\lambda^n\right| \le \frac{C_A}{\log^A N} \qquad A>1 \qquad \forall \lambda \in \mathbb{T}$$

Davenport estimate