Theoretical aspects of the finite element method in solving fractional wave equations with nonsmooth data

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We aim to focus on the finite element method (FEM) based on a corrected formula designed for solving the following fractional wave equation

$$\begin{cases} {}_0^C\mathrm{D}_t^\alpha u + \mathcal{L}u = f(t), & 0 \le t \le T, \\ u(0) = u_0, & u'(0) = u_1, \end{cases}$$

where ${}_0^C\mathrm{D}_t^\alpha$ is the Caputo fractional derivative with respect to t, i.e.

$${}_0^C\mathrm{D}_t^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}\, u''(s)\, \mathrm{d}s, \qquad \alpha \in (1,2),$$

in which $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, $\mathcal{L} = -\Delta$ that is the Laplacian with the definition domain $D(\mathcal{L}) = H_0^1(\Omega) \cap H^2(\Omega)$, and $u_0, u_1 \in L^2(\Omega)$ are some given functions. This model problem has applications in modeling processes characterized by non-local effects such as anomalous diffusion and infiltration. It highlights the importance of maintaining temporal accuracy when dealing with nonsmooth data. Traditional numerical schemes often struggle to maintain the desired temporal order of convergence in such scenarios. The complexities of fractional wave equations, especially when nonsmooth initial or source data are involved, worsen these challenges, leading to a breakdown in the accuracy and efficiency of classic methods. Our approach introduces a corrected scheme within the FEM framework that is specifically designed to preserve temporal order, even in the presence of nonsmooth data. Leveraging time discretization techniques with an enhanced correcting initial step, our method overcomes the limitations of conventional approaches. The theoretical foundation of this method is grounded in the Laplace transform method. We present analysis of the method's performance, including error estimates. Finally, we confirm the theoretical results using a numerical example.

References:

[1] M. Ramezani, R. Mokhtari, Y. Yan, Correction of a high-order numerical method for approximating time-fractional wave equation. Journal of Scientific Computing, 100(3), 71, (2024).

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