# Sparse source detection for advection-diffusion problems

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This talk presents an application of sparse optimization techniques for advection-diffusion problems. The advection-diffusion equation is suitable for describing the dispersion of a pollutant concentration $u : (0,T) \times \Omega \to \mathbb{R}$:

$$
\begin{aligned}
u_t - \kappa \Delta u + \mathbf{v} \cdot \nabla u &= 0 && \text{in } (0,T) \times \Omega, \\
\kappa \nabla u \cdot \eta &= 0 && \text{in } (0,T) \times \Gamma_N, \\
u &= 0 && \text{in } (0,T) \times \Gamma_D, \\
u(0,\cdot) &= m && \text{in } \Omega.
\end{aligned}
$$

While the forward problem $\mathcal{F}(m) = u$ describes the evolution of $u$ over time, the aim of the inverse problem is to predict the source of the pollutant based on several discrete noisy sensor measurements $d$. Since the release of pollutants under real conditions is very localized, an approach was chosen that promotes sparsity in $m$. To model this, an optimization problem is formulated for the initial value $m$

$$
\min_{m \in \mathcal{M}} \frac{1}{2} ||\mathcal{F}(m) - d||^2_{\Gamma^{-1}_{\text{noise}}} + \alpha \, \|m\|_{\mathcal{M}(\Omega)} \, .
$$

The regularization requires, that $m$ is in the space of regular Borel measures $\mathcal{M}(\Omega)$ understood as a dual space of the continuous functions $C(\Omega)$:

$$
\|m\|_{\mathcal{M}(\Omega)} = \sup\{\langle m, \varphi \rangle : \varphi \in C(\overline{\Omega}), \|\varphi\|_{C(\overline{\Omega})} = 1\}.
$$

For the numerical solution, the linear advection-diffusion equation and the corresponding adjoint equation is discretized by standard stabilized continuous Galerkin FEM methods. For the solution of the inverse problem a Primal-Dual-Active-Point strategy (PDAP) is used. The talk is mainly intended to discuss the near real-time capabilities of this method and its applicability for the protection of critical infrastructures (KRITIS).

References:

[1] https://doi.org/10.1051/cocv/2021042

[2] https://iopscience.iop.org/article/10.1088/1361-6420/ad2cf8

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