A well-balanced finite element scheme based on monolithic convex limiting

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To obtain accurate approximate solutions, the used numerical scheme have to be consistent with the solved problem in an appropriate sense. One possible requirement is that the scheme preserves some simple solutions (equilibria) of the continuous problem, which is particularly important when inhomogeneous balance laws are considered. Discretizations that provide such consistency are called well balanced in the literature. We will consider this property in the context of finite element discretizations of steady convection-diffusion-reaction equations.

Recently, a well-balanced and positivity-preserving finite element scheme was developed by Hajduk (2022) for the inhomogeneous shallow water equations using the framework of algebraic flux correction. The monolithic convex limiting (MCL) algorithm presented in Hajduk (2022) and Hajduk, Kuzmin (2022) incorporates discretized bathymetry gradients into the numerical fluxes and intermediate states of the spatial semi-discretization. In the present contribution, we show that the source term of a scalar convection-diffusion-reaction problem can be treated similarly. In particular, we define numerical fluxes that ensure consistency of the well-balanced MCL approximation with a linear steady state. Using a convex decomposition into intermediate states, we enforce positivity preservation, as well as local and global discrete maximum principles. Moreover, the design of the method enables to prove that the nonlinear discrete problem possesses a solution. Numerical results illustrate potential benefits of well-balanced flux limiting in the scalar case. Details can be found in Knobloch, Kuzmin, Jha (2024).

References:

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